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# A four-dimensional approach to quantum field theories

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**ABSTRACT:** I present a novel Four-Dimensional Regularization/Renormalization approach (FDR) to ultraviolet divergences in field theories which can be interpreted as a natural separation between physical and non physical degrees of freedom. Based on the observation that, in some cases, infinities can be reabsorbed into the vacuum expectation value of the fields, a new type of four-dimensional, gauge invariant and cutoff independent loop integral is introduced, which reproduces the correct ABJ anomaly and does not require changes in the definition of  $\gamma_5$ . Finally, I comment on a possible interpretation of non-renormalizable theories in the context of the proposed procedure, and show how FDR can also be used to regularize infrared and collinear divergences.

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## 1 Introduction

The study of the ultraviolet (UV) infinities occurring in quantum field theories has a long story [1–4]. The commonly accepted interpretation is that they occur in intermediate steps of loop calculations, but have no physical meaning, being reabsorbable into a redefinition of the parameters of the theory, which should therefore be readjusted order by order. The proof that this can be carried out in a way consistent with the symmetries of the theory is at the base of the renormalization program, and requires to regularize the divergences by means of a regulator, showing that the dependence on the regulator drop out in physical observables. A mathematical proof of the feasibility of this approach is due to the Bogoliubov, Parasiuk [5], Hepp [6] and Zimmermann [7] (BPHZ). In the BPHZ renormalization scheme, the divergent Green functions are Taylor expanded up to the order needed to reach convergent integrals, and the renormalization conditions are imposed directly on the finite terms of the expansion. A key ingredient for the success of this procedure is showing

that overlapping divergences, which cannot be compensated by local counterterms, cancel. When applied to gauge theories, BPHZ may brake gauge invariance in the intermediate steps, and is technically quite involved. Therefore, other alternatives, more manageable from a technical point of view, have been proposed, such as Pauli Villars [8], or Speer regulators [9], which, however, may lead to gauge dependent results. This is not yet considered to be a serious drawback, since the missing pieces could be reinserted back by enforcing the Ward-Slavnov-Taylor identities of the theory. The situation can then be summarized as follows: the symmetry of the theory is a guideline to prove the correctness of the result.

In general, restoring gauge invariance may be technically difficult and error-prone, so when Dimensional Regularization (DR) emerged [10], which automatically respects gauge symmetries, the proof of the renormalizability of the Yang-Mills theories, with and without spontaneous symmetry breaking, could be carried out, and their predictive power started being exploited.

However, there are theories which diverge so badly that new infinities are generated order by order in the perturbative expansion, which cannot be reabsorbed into the Lagrangian, at least with a finite number of counterterms. Such theories are called non-renormalizable, and are commonly interpreted as effective ones, the fundamental truth being still to be unveiled.

In this paper, I present an interpretation of the UV infinities as unphysical degrees of freedom, which have to decouple. Elaborating on its consequences, I introduce an approach, dubbed Four-Dimensional Regularization/Renormalization (FDR), in which a key role is played by the definition of a four-dimensional, cut-off free, gauge invariant integral, which allows one to directly compute renormalized Green Functions with no need of reabsorbing infinities into the Lagrangian. When applied to non-renormalizable theories, FDR leads to a possible interpretation in which they could acquire some predictivity.

The outline of the paper is as follows. Section 2 explains how UV divergences and physical degrees of freedom can be disentangled. In sections 3, 4 and 5 the FDR integral is introduced, and the one-loop and two-loop cases discussed in detail. In section 6 I show how the ABJ anomaly is naturally predicted by FDR, and comment on the role of  $\gamma_5$ . The FDR renormalization program is discussed in section 7, together with the aforementioned interpretation of the non-renormalizable theories. Finally, the use of FDR to regularize infrared and collinear divergences is presented in section 8.

## 2 The FDR approach to UV divergences

A simple redefinition of the vacuum may reabsorb some of the UV infinities occurring at large values of the integration momentum. To illustrate this phenomenon, I start with the simplest possible example, namely a scalar theory with cubic interaction <sup>1</sup>

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{M^2}{2}\Phi^2 - M\frac{\lambda}{3!}\Phi^3. \quad (2.1)$$

The tadpole contribution generated at one-loop is usually compensated by introducing an *ad hoc* counterterm in the Lagrangian. Instead of following this procedure, I perform a

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<sup>1</sup>To be considered part of a more complete theory.

$$\begin{aligned}
\text{---}\overrightarrow{p}\text{---} &= \frac{i}{p^2 - M^2}, & \text{---}\text{---}\text{---} &= -i\lambda M, \\
\text{---}\otimes\text{---} &= -i\left(M^2 v + \frac{\lambda M}{2}v^2\right), & \text{---}\otimes\text{---}\text{---} &= -i\lambda M v.
\end{aligned}$$

**Figure 1.** Feynman rules for the  $\lambda\Phi^3$  theory (top) and extra vertices generated by the shift  $\Phi \rightarrow \Phi + v$  (bottom).

global field shift <sup>2</sup>

$$\Phi \rightarrow \Phi + v, \quad (2.2)$$

which reproduces the usual Feynman rules plus extra ones, as drawn in figure 1. The only divergent diagrams at one-loop are the tadpole  $T$  and the 2-point function  $i\Sigma$

$$T = \frac{\lambda M}{2}I_2, \quad i\Sigma = \frac{\lambda^2 M^2}{2}I_0. \quad (2.3)$$

The one-loop scalar integrals <sup>3</sup>

$$I_2 = \mu_R^{-\epsilon} \int d^n q \frac{1}{D}, \quad I_0 = \mu_R^{-\epsilon} \int d^n q \frac{1}{D D_p}, \quad (2.4)$$

with

$$D = (q^2 - M^2), \quad D_p = ((q + p)^2 - M^2), \quad (2.5)$$

can be computed in  $n = 4 + \epsilon$  dimensions as

$$\begin{aligned}
I_2 &= \lim_{\mu \rightarrow 0} \mu_R^{-\epsilon} \int d^n q \frac{1}{\bar{D}} \\
I_0 &= \lim_{\mu \rightarrow 0} \mu_R^{-\epsilon} \int d^n q \frac{1}{\bar{D} \bar{D}_p},
\end{aligned} \quad (2.6)$$

where a small extra mass  $\mu$  has been introduced in the propagators

$$\bar{D} = D - \mu^2, \quad \bar{D}_p = D_p - \mu^2. \quad (2.7)$$

That allows one to use the partial fraction identities

$$\begin{aligned}
\frac{1}{\bar{D}} &= \frac{1}{\bar{q}^2} \left(1 + \frac{M^2}{\bar{D}}\right), \\
\frac{1}{\bar{D}_p} &= \frac{1}{\bar{q}^2} \left(1 + \frac{d(q)}{\bar{D}_p}\right),
\end{aligned} \quad (2.8)$$

<sup>2</sup>Assuming no normal ordering for the Lagrangian.

<sup>3</sup> $\mu_R$  denotes the renormalization scale.

$$\text{---}\bigotimes\text{---} + \text{---}\bigcirc\text{---} = 0$$

**Figure 2.** The no-tadpole condition on the shifted  $\lambda\Phi^3$  scalar theory.

where

$$\bar{q}^2 = q^2 - \mu^2, \quad d(q) = M^2 - p^2 - 2(q \cdot q), \quad (2.9)$$

to rewrite the denominators as

$$\frac{1}{\bar{D}} = \frac{1}{\bar{q}^2} + \frac{M^2}{\bar{q}^4} + \frac{M^4}{\bar{D}\bar{q}^4} \quad (2.10)$$

and

$$\frac{1}{\bar{D}\bar{D}_p} = \frac{1}{\bar{q}^4} + \frac{d(q)}{\bar{q}^4\bar{D}_p} + \frac{M^2}{\bar{q}^2\bar{D}\bar{D}_p}. \quad (2.11)$$

In this way, the UV divergences are moved to the first two terms of eq. 2.10 and to the first one of eq. 2.11<sup>4</sup>, and since

$$\lim_{\mu \rightarrow 0} \mu_R^{-\epsilon} \int d^n q \frac{1}{\bar{q}^2} = 0, \quad (2.12)$$

only the logarithmically divergent integral

$$\mu_R^{-\epsilon} \int d^n q \frac{1}{\bar{q}^4} \equiv i I_{inf} \quad (2.13)$$

remains. The divergent parts are then computed as

$$T|_{inf} = i\lambda \frac{M^3}{2} I_{inf}, \quad i\Sigma|_{inf} = i\lambda \frac{M^2}{2} I_{inf}, \quad (2.14)$$

in terms of a common integral which no longer depends on any physical scale. One can now define the  $\Phi$  vacuum in such a way that no tadpoles occur, as in figure 2, which fixes  $v$  to the value

$$v = \lambda M \frac{I_{inf}}{2} + \mathcal{O}(\lambda^3). \quad (2.15)$$

When including the extra vertex in the computation of the  $\Phi$  self energy  $\bar{\Sigma}$ , as in figure 3, this is just what one needs to cancel its UV behavior

$$i\bar{\Sigma}|_{inf} = 0. \quad (2.16)$$

Notice that both divergences have been removed by fixing just one parameter<sup>5</sup>, while the

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<sup>4</sup>Without  $\mu$  the remaining terms would generate infrared divergences.

$$i\bar{\Sigma} = \text{---} \bigcirc \text{---} + \text{---} \bigotimes \text{---}$$

**Figure 3.** The complete  $\Phi$  self-energy in the shifted  $\lambda\Phi^3$  theory.

$$\begin{aligned} \overrightarrow{\text{---} p \text{---}} &= \frac{i}{p^2 - M^2}, & \text{X} &= -i\lambda, \\ \text{---} \bigotimes \text{---} &= -i\lambda v, & \text{---} \bigotimes \text{---} &= -i\frac{\lambda v^2}{2}, & \text{---} \bigotimes &= -i\left(M^2 v + \frac{\lambda}{6} v^3\right). \end{aligned}$$

**Figure 4.** Feynman rules for the  $\lambda\Phi^4$  theory (top) and extra vertices generated by the shift  $\Phi \rightarrow \Phi + v$  (bottom).

$$\text{---} \blacksquare \text{---} = \text{---} \bigotimes \text{---} + \text{---} \bigotimes \text{---} = \frac{i\lambda^2}{6} \frac{v^3}{M^2}$$

**Figure 5.** A useful identity for the shifted  $\lambda\Phi^4$  theory.

traditional approach would have required two counterterms. As will be explained later, the dependence on  $\mu$  obtained by integrating in four dimensions the left over terms in eqs. 2.10 and 2.11 can only be logarithmic, and can be traded for the usual dependence on  $\mu_R$ .

In a scalar theory with quartic interaction

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{M^2}{2}\Phi^2 - \frac{\lambda}{4!}\Phi^4, \quad (2.17)$$

no tadpole is generated at one-loop. Nevertheless, one can still reabsorb the mass renormalization into the vacuum expectation value of  $\Phi$ . The same shift of eq. 2.2 generates now the extra Feynman rules given in figure 4. Due to the identity of figure 5, 3-point vertices and tadpoles do not occur at  $\mathcal{O}(\lambda)$ , thus the  $\Phi$  self-energy is simply obtained by summing the one-loop diagram and the 2-point extra vertex, as in figure 6, giving

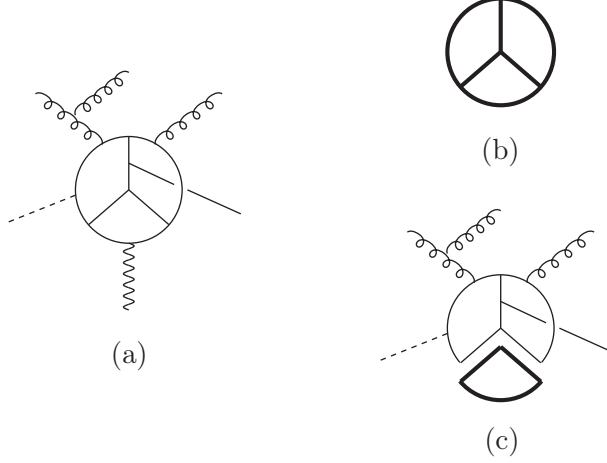
$$i\bar{\Sigma} = i\frac{\lambda}{2}(M^2 I_{inf} - v^2). \quad (2.18)$$

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<sup>5</sup>In agreement with ref. [11].

$$i\bar{\Sigma} = \text{---}\text{---}\text{---}\text{---} + \text{---}\text{---}\text{---}\text{---}$$

**Figure 6.** The complete  $\Phi$  self-energy in the shifted  $\lambda\Phi^4$  scalar theory.



**Figure 7.** Generic diagram contributing to the interaction (a). *Vacuum diagrams* generated when all integration momenta are large (b) and when one sub-loop integration momentum goes to infinity (c).

The value  $v^2 = M^2 I_{inf}$  is such that no mass renormalization is needed. This solution is acceptable perturbatively, but it does not represent a correction  $\mathcal{O}(\lambda)$  around  $\langle \phi \rangle = 0$ , as in the previous case. One can impose vacuum stability at higher orders by spontaneously breaking the symmetry with the condition

$$M^2 + \frac{\lambda v^2}{6} = 0, \quad (2.19)$$

and repeating the exercise with this new vacuum.

The lesson to be learnt from these examples is that unphysical degrees of freedom occurring in UV divergent loop integrals decouple once observable physics is described relatively to the vacuum. In addition, a parametrization as in eqs. 2.10 and 2.11, in which the UV part is moved to an integral  $I_{inf}$ , which does not depend on any physical scale (and therefore very much looks like a *vacuum diagram* or a *vacuum bubble*), appears to be a quite natural tool to achieve, in practice, a separation between physical observables and vacuum fluctuations. This idea is pictorially illustrated in figure 7, which represents a generic diagram contributing to a connected Green function. When the loop (sub)-integration momenta become large, all momenta attached to it and all internal masses become negligible and the (sub)-diagram effectively behaves like a *vacuum bubble*, as shown in figures 7 (b) and (c), and decouples. Such large loop momentum states (which I label *vacuum configurations*), being universal, do not belong to the interaction any more and

should be removed. In this paper, I adopt a very pragmatic approach and show that the integral over the terms which remain after subtracting these unphysical modes owns all good properties one wishes to perform practical calculations, namely it is four-dimensional, gauge invariant and independent on any cutoff, as will be explained in the next section. In this respect, one could just *define* differently the loop integrals, so that the mechanism nature uses to wipe the infinities out, either by reabsorption into the vacuum, as just shown, or via renormalization, becomes less relevant.

### 3 The FDR integral

The first problem is how to recognize, classify and subtract from any given diagram the unphysical large loop momentum configurations, and how to deal with them. Eqs. 2.10 and 2.11 serve as a guideline. First of all, a convenient parametrization is needed, in terms of an arbitrary scale, called  $\mu$ , compared to which the loop momenta are considered to be large. Thus, one expects the high frequencies to decouple in the limit  $\mu \rightarrow 0$ . Secondly, one requires independence on the UV cutoff, which means no left-over dependence on  $\mu$  in physical quantities. With all of that in mind, I represent *vacuum configurations* as  $\ell$ -loop integrals which only depend on the unphysical scale  $\mu$ . A rank-r one-loop example is

$$\mu_R^{-\epsilon} \int d^n q \frac{q_{\alpha_1} \cdots q_{\alpha_r}}{(q^2 - \mu^2)^j}, \quad (3.1)$$

and a scalar two-loop case reads

$$\mu_R^{-2\epsilon} \int d^n q_1 d^n q_2 \frac{1}{(q_1^2 - \mu^2)^{j_1} (q_2^2 - \mu^2)^{j_2} ((q_1 + q_2)^2 - \mu^2)^{j_3}}. \quad (3.2)$$

Note that, according to the values of  $j$ ,  $j_1$ ,  $j_2$  and  $j_3$ , they may be or may be not UV divergent. As a matter of definition, I call *vacuum integrands* (or *vacuum terms*) all integrands such as those appearing in eqs. 3.1 and 3.2 which only depend on the unphysical scale  $\mu$ <sup>6</sup>. Let now  $I_{\ell-loop}^{\text{DR}}$  a representative  $\ell$ -loop diagram contributing to a Green function computed in DR. Then

$$I_{\ell-loop}^{\text{DR}} = \mu_R^{-\ell\epsilon} \int \prod_{i=1}^{\ell} d^n q_i J(\{q_\alpha, q^2, \not{q}\}), \quad (3.3)$$

where  $\{q_\alpha, q^2, \not{q}\}$  symbolically denotes the set of loop-integration variables upon which the integrand  $J$  depends, where I distinguish among tensor like structures, denoted by  $q_\alpha$ , self contracted loop-momenta  $q^2$  and contractions with  $\gamma$  matrices,  $\not{q}$ . Clearly, as done in eq. 2.6<sup>7</sup>,

$$I_{\ell-loop}^{\text{DR}} = \lim_{\mu \rightarrow 0} \mu_R^{-\ell\epsilon} \int \prod_{i=1}^{\ell} d^n q_i J(\{q_\alpha, q^2 - \mu^2, \not{q} - \mu\}), \quad (3.4)$$

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<sup>6</sup>Abusing a bit the language, I define finite, logarithmic and quadratic divergent vacuum integrands those generating finite, globally logarithmic and quadratic divergent integrals, etc.

<sup>7</sup>I assume here no infrared or collinear divergences in  $I_{\ell-loop}^{\text{DR}}$ . Such cases are discussed in section 8.



where it is understood that the replacements in the integrand affect not only boson and fermion propagators, which gets modified as follows <sup>8</sup>

$$\begin{aligned}\frac{1}{(q+p)^2 - m^2} &\rightarrow \frac{1}{(q+p)^2 - m^2 - \mu^2} \\ \frac{1}{\not{q} + \not{p} - m} &\rightarrow \frac{1}{\not{q} + \not{p} - m - \mu},\end{aligned}\tag{3.5}$$

but also numerators, as required by gauge invariance. In the following,  $\mu$  will often appear in combination with the propagators  $D_i$ ,  $q^2$  and  $\not{q}$ , thus I use the same notation introduced in the previous section, namely

$$\bar{D}_i \equiv D_i - \mu^2, \quad \bar{q}^2 \equiv q^2 - \mu^2, \quad \bar{\not{q}} \equiv \not{q} - \mu.\tag{3.6}$$

Eq. 3.4 is the place where the unphysical scale  $\mu$  which separates the large-loop vacuum configurations is introduced. Suppose now to use eq. 2.8 to split the integrand  $J$  into a part containing all possible divergent vacuum integrands ( $J_V$ ) and a piece integrable in four dimensions

$$J(\{q_\alpha, \bar{q}^2, \bar{\not{q}}\}) = J_V(\{q_\alpha, \bar{q}^2, \bar{\not{q}}\}) + J_F(\{q_\alpha, \bar{q}^2, \bar{\not{q}}\}).\tag{3.7}$$

In order to keep gauge cancellations, it is important to perform this separation globally on  $\bar{q}^2$  and  $\bar{\not{q}}$ , meaning that their  $\mu$  parts should not be treated differently from  $q^2$  and  $\not{q}$  <sup>9</sup>. The FDR integral is then defined as

$$I_{\ell-loop}^{\text{FDR}} = \int \prod_{i=1}^{\ell} [d^4 q_i] J(\{q_\alpha, \bar{q}^2, \bar{\not{q}}\}) \equiv \lim_{\mu \rightarrow 0} \int \prod_{i=1}^{\ell} d^4 q_i J_F(\{q_\alpha, \bar{q}^2, \bar{\not{q}}\}) \Big|_{\mu=\mu_R},\tag{3.8}$$

where the symbol  $\int [d^4 q]$  means:

1. use eq. 2.8 to move all divergences in vacuum integrands, treating  $\bar{q}^2$  and  $\bar{\not{q}}$  globally;
2. drop all divergent vacuum terms from the integrand;
3. integrate over  $d^4 q$ ;
4. take the limit  $\mu \rightarrow 0$ , until a logarithmic dependence on  $\mu$  is reached;
5. compute the result in  $\mu = \mu_R$ .

The FDR integral is a physical quantity in which all high frequencies giving rise to unphysical vacuum configurations either do not contribute or are fully subtracted. Consider, in fact, the connection of  $I_{\ell-loop}^{\text{FDR}}$  with the original integral of eq. 3.4

$$I_{\ell-loop}^{\text{FDR}} = I_{\ell-loop}^{\text{DR}} - \lim_{\mu \rightarrow 0} \mu_R^{-\ell\epsilon} \int \prod_{i=1}^{\ell} d^n q_i J_V(\{q_\alpha, \bar{q}^2, \bar{\not{q}}\}) \Big|_{\mu=\mu_R}.\tag{3.9}$$

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<sup>8</sup>A small negative imaginary part of  $\mu$  generates the correct  $+i0$  propagator prescription as well.

<sup>9</sup>I call this a *global treatment* of  $\bar{q}^2$  and  $\bar{\not{q}}$ .

Since the only available scale is  $\mu$ , the contribution to  $J_V$  of polynomially divergent vacuum integrands vanishes, when  $\mu \rightarrow 0$ . Thus, large loop polynomially divergent vacuum configurations immediately decouple. Logarithmically divergent integrands give instead a contribution of the form

$$K + \sum_{i=1}^{\ell} a_i \ln^i(\mu/\mu_R), \quad (3.10)$$

where  $K$  contains infinities and constant terms. Since  $I_{\ell-loop}^{\text{DR}}$  is independent on  $\mu$ , both sides of 3.9 have the same  $\mu$  dependence given in eq. 3.10, before computing them at  $\mu = \mu_R$ . The powers of  $\ln(\mu)$ , being generated by the low energy regime of the integration momenta, do not decouple and should be moved to the physical part  $I_{\ell-loop}^{\text{FDR}}$ . Then one can take the formal limit  $\mu \rightarrow 0$  in it<sup>10</sup> by trading  $\mu$  for  $\mu_R$ . Differently stated, the point  $\mu = \mu_R$ <sup>11</sup> is such that also the logarithmic divergent vacuum bubbles completely decouple. As for the high frequencies of finite vacuum bubbles, they naturally give vanishing contributions at large values of the loop momenta, by power counting. If one subtracts them, unphysical arbitrary powers of  $1/\mu$  are generated in the physical part, that should be compensated by moving back to it analogous poles created in  $J_V$ .

Eq. 3.8 defines a multi-loop integral with all good properties one expects. It is finite in four dimensions, cut-off independent and invariant under any shift of the integration variables. The latter property follows from the fact that it can be also defined as the difference of two DR integrals, as in eq. 3.9. It respects gauge invariance by construction, because of the shift invariance and of the *global treatment* of  $\vec{q}^2$  and  $\vec{q}$ . This means that properties such as

$$\begin{aligned} \int \prod_{i=1}^{\ell} [d^4 q_i] J(\{q_{\alpha}, \vec{q}^2, \vec{q}\}) &= \int \prod_{i=1}^{\ell} [d^4 q_i] J(\{q_{\alpha}, \vec{q}^2, \vec{q}\}) \frac{\vec{q}_j^2}{\vec{q}_j^2 - M^2} \\ &- \int \prod_{i=1}^{\ell} [d^4 q_i] J(\{q_{\alpha}, \vec{q}^2, \vec{q}\}) \frac{M^2}{\vec{q}_j^2 - M^2} \quad \forall q_j \in \{q_{\alpha}, \vec{q}^2, \vec{q}\} \end{aligned} \quad (3.11)$$

are guaranteed.

## 4 One-loop examples

### 4.1 Scalar integrals not depending on external momenta

I start with a simple logarithmically divergent integral with no external scale

$$\int [d^4 q] \frac{1}{D^2}, \quad (4.1)$$

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<sup>10</sup>This is important because no dependence on the initial cutoff is allowed in the physical part.

<sup>11</sup>In which eq. 3.10 reduces to  $K$ .

with  $\bar{D}$  given in eq. 2.7. Eq. 2.11 with  $p = 0$  can be used to extract the divergent vacuum integrands, which, from now on, I write inside square brackets

$$\frac{1}{\bar{D}^2} = \left[ \frac{1}{\bar{q}^4} \right] + M^2 \left( \frac{1}{\bar{D}^2 \bar{q}^2} + \frac{1}{\bar{D} \bar{q}^4} \right). \quad (4.2)$$

Then <sup>12</sup>

$$I_0^{\text{FDR}} = \int [d^4 q] \frac{1}{\bar{D}^2} \equiv \lim_{\mu \rightarrow 0} M^2 \int d^4 q \left( \frac{1}{\bar{D}^2 \bar{q}^2} + \frac{1}{\bar{D} \bar{q}^4} \right) \Big|_{\mu=\mu_R} = -i\pi^2 \ln \frac{M^2}{\mu_R^2}. \quad (4.3)$$

In an analogous way, eq. 2.10 gives <sup>13</sup>

$$I_2^{\text{FDR}} = \int [d^4 q] \frac{1}{\bar{D}} \equiv \lim_{\mu \rightarrow 0} M^4 \int d^4 q \frac{1}{\bar{D} \bar{q}^4} \Big|_{\mu=\mu_R} = -i\pi^2 M^2 \left( \ln \frac{M^2}{\mu_R^2} - 1 \right). \quad (4.4)$$

Note also that, by definition of FDR

$$\int [d^4 q] (\bar{q}^2)^j = 0 \quad \forall j \geq -2. \quad (4.5)$$

Eqs. 4.3 and 4.4 coincide with the corresponding dimensionally regulated expressions in the  $\overline{\text{MS}}$  scheme. This is because only one logarithmically divergent scalar integrand exists at one-loop, which can contribute to  $J_V$  in eq. 3.9. Thus, a correspondence exists

$$\frac{1}{\epsilon} + \text{UC subtraction after integration} \leftrightarrow \frac{1}{\bar{q}^4} \text{ subtraction before integration.}$$

Universal Constants (UC) also appear in the l.h.s. because the full integrand is subtracted in FDR.

## 4.2 Shifting the integration momentum

Although shift invariance is guaranteed by construction, it is instructive to verify this property in simple cases. Let me consider, for example, the shifted versions of  $I_0^{\text{FDR}}$  and  $I_2^{\text{FDR}}$

$$\begin{aligned} I_{0p}^{\text{FDR}} &= \int [d^4 q] \frac{1}{\bar{D}_p^2} \\ I_{2p}^{\text{FDR}} &= \int [d^4 q] \frac{1}{\bar{D}_p}, \end{aligned} \quad (4.6)$$

where  $\bar{D}_p = (q + p)^2 - M^2 - \mu^2$  and  $p$  is an arbitrary 4-vector. By iteratively using eq. 2.8 one rewrites

$$\begin{aligned} \frac{1}{\bar{D}_p^2} &= \left[ \frac{1}{\bar{q}^4} \right] + \frac{d(q)}{\bar{q}^4 \bar{D}_p} + \frac{d(q)}{\bar{q}^2 \bar{D}_p^2} \\ \frac{1}{\bar{D}_p} &= \left[ \frac{1}{\bar{q}^2} + \frac{d(q)}{\bar{q}^4} + \frac{(d_1 \cdot q)^2}{\bar{q}^6} \right] + \frac{d_0^2 + 2d_0(d_1 \cdot q)}{\bar{q}^4 \bar{D}_p} + \frac{d(q)(d_1 \cdot q)^2}{\bar{q}^6 \bar{D}_p}, \end{aligned} \quad (4.7)$$

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<sup>12</sup>See appendix A.

<sup>13</sup>See again appendix A.

with

$$d_0 = M^2 - p^2, \quad d_1^\mu = -2p^\mu, \quad (4.8)$$

which implies, by definition

$$\begin{aligned} \int [d^4 q] \frac{1}{\bar{D}_p^2} &\equiv \lim_{\mu \rightarrow 0} \int d^4 q \left( \frac{d(q)}{\bar{q}^4 \bar{D}_p} + \frac{d(q)}{\bar{q}^2 \bar{D}_p^2} \right) \Big|_{\mu=\mu_R}, \\ \int [d^4 q] \frac{1}{\bar{D}_p} &\equiv \lim_{\mu \rightarrow 0} \int d^4 q \left( \frac{d_0^2 + 2d_0(d_1 \cdot q)}{\bar{q}^4 \bar{D}_p} + \frac{d(q)(d_1 \cdot q)^2}{\bar{q}^6 \bar{D}_p} \right) \Big|_{\mu=\mu_R}. \end{aligned} \quad (4.9)$$

Integrating the second of eqs. 4.9 is a bit involved, but there are not surprises:

$$\begin{aligned} I_{0p}^{\text{FDR}} &= I_0^{\text{FDR}} \\ I_{2p}^{\text{FDR}} &= I_2^{\text{FDR}}. \end{aligned} \quad (4.10)$$

### 4.3 Scalar integrals depending on external momenta

Equipped with the previous results, it is straightforward to show that

$$\begin{aligned} \int [d^4 q] \frac{1}{\bar{D}_0 \bar{D}_1} &\equiv \lim_{\mu \rightarrow 0} \int d^4 q \left( \frac{d(q)}{\bar{q}^4 \bar{D}_1} + m_0^2 \frac{1}{\bar{q}^4 \bar{D}_0} + m_0^2 \frac{d(q)}{\bar{q}^4 \bar{D}_0 \bar{D}_1} \right) \Big|_{\mu=\mu_R} \\ &= -i\pi^2 \int_0^1 d\alpha \ln \frac{\chi(\alpha)}{\mu_R^2}, \end{aligned} \quad (4.11)$$

where

$$\bar{D}_0 = q^2 - m_0^2 - \mu^2, \quad \bar{D}_1 = (q+p)^2 - m_1^2 - \mu^2, \quad d(q) = m^2 - p^2 - 2(p \cdot q), \quad (4.12)$$

and

$$\chi(\alpha) = m_0^2 \alpha + m_1^2 (1 - \alpha) - p^2 \alpha (1 - \alpha). \quad (4.13)$$

Eq. 4.11 again coincides with the  $\overline{\text{MS}}$  result. Notice also that, due to shift invariance,

$$\int [d^4 q] \frac{1}{\bar{D}_0 \bar{D}_1} = \int_0^1 d\alpha \int [d^4 q] \frac{1}{[\bar{q}^2 - \chi(\alpha)]^2}. \quad (4.14)$$

### 4.4 Tensor integrals

From the identities

$$\begin{aligned} \frac{1}{\bar{D}^2} &= \left[ \frac{1}{\bar{q}^4} + 2 \frac{M^2}{\bar{q}^6} \right] + M^4 \left( \frac{2}{\bar{D} \bar{q}^6} + \frac{1}{\bar{D}^2 \bar{q}^4} \right), \\ \frac{1}{\bar{D}^3} &= \left[ \frac{1}{\bar{q}^6} \right] + M^2 \left( \frac{1}{\bar{D}^3 \bar{q}^2} + \frac{1}{\bar{D}^2 \bar{q}^4} + \frac{1}{\bar{D} \bar{q}^6} \right), \\ \frac{1}{\bar{D}^4} &= \left[ \frac{1}{\bar{q}^8} \right] + M^2 \left( \frac{1}{\bar{D}^4 \bar{q}^2} + \frac{1}{\bar{D}^3 \bar{q}^4} + \frac{1}{\bar{D}^2 \bar{q}^6} + \frac{1}{\bar{D} \bar{q}^8} \right), \\ \frac{1}{\bar{D}^3} &= \left[ \frac{1}{\bar{q}^6} + 3 \frac{M^2}{\bar{q}^8} \right] + M^4 \left( \frac{3}{\bar{D} \bar{q}^8} + \frac{2}{\bar{D}^2 \bar{q}^6} + \frac{1}{\bar{D}^3 \bar{q}^4} \right), \end{aligned} \quad (4.15)$$

with  $\bar{D}$  defined in eq. 2.7, one obtains

$$\begin{aligned}
\int [d^4 q] \frac{q^\alpha q^\beta}{\bar{D}^2} &= M^4 \lim_{\mu \rightarrow 0} \int d^4 q q^\alpha q^\beta \left( \frac{2}{\bar{D} \bar{q}^6} + \frac{1}{\bar{D}^2 \bar{q}^4} \right) \Big|_{\mu=\mu_R}, \\
\int [d^4 q] \frac{q^\alpha q^\beta}{\bar{D}^3} &= M^2 \lim_{\mu \rightarrow 0} \int d^4 q q^\alpha q^\beta \left( \frac{1}{\bar{D}^3 \bar{q}^2} + \frac{1}{\bar{D}^2 \bar{q}^4} + \frac{1}{\bar{D} \bar{q}^6} \right) \Big|_{\mu=\mu_R}, \\
\int [d^4 q] \frac{q^\alpha q^\beta q^\gamma q^\delta}{\bar{D}^4} &= M^2 \lim_{\mu \rightarrow 0} \int d^4 q q^\alpha q^\beta q^\gamma q^\delta \left( \frac{1}{\bar{D}^4 \bar{q}^2} + \frac{1}{\bar{D}^3 \bar{q}^4} + \frac{1}{\bar{D}^2 \bar{q}^6} + \frac{1}{\bar{D} \bar{q}^8} \right) \Big|_{\mu=\mu_R}, \\
\int [d^4 q] \frac{q^\alpha q^\beta q^\gamma q^\delta}{\bar{D}^3} &= M^4 \lim_{\mu \rightarrow 0} \int d^4 q q^\alpha q^\beta q^\gamma q^\delta \left( \frac{3}{\bar{D} \bar{q}^8} + \frac{2}{\bar{D}^2 \bar{q}^6} + \frac{1}{\bar{D}^3 \bar{q}^4} \right) \Big|_{\mu=\mu_R}. \quad (4.16)
\end{aligned}$$

Given the four-dimensional definition, one can replace

$$\begin{aligned}
q^\alpha q^\beta &\rightarrow \frac{q^2}{4} g^{\alpha\beta} \\
q^\alpha q^\beta q^\gamma q^\delta &\rightarrow \frac{q^4}{24} g^{\alpha\beta\gamma\delta} \\
g^{\alpha\beta\gamma\delta} &\equiv (g^{\alpha\beta} g^{\gamma\delta} + g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma}) \quad (4.17)
\end{aligned}$$

in eq. 4.16 and compute

$$\begin{aligned}
\int [d^4 q] \frac{q^\alpha q^\beta}{\bar{D}^2} &= \frac{g^{\alpha\beta}}{2} I_2^{\text{FDR}}, \\
\int [d^4 q] \frac{q^\alpha q^\beta}{\bar{D}^3} &= \frac{g^{\alpha\beta}}{4} I_0^{\text{FDR}}, \\
\int [d^4 q] \frac{q^\alpha q^\beta q^\gamma q^\delta}{\bar{D}^4} &= \frac{g^{\alpha\beta\gamma\delta}}{24} I_0^{\text{FDR}}, \\
\int [d^4 q] \frac{q^\alpha q^\beta q^\gamma q^\delta}{\bar{D}^3} &= \frac{g^{\alpha\beta\gamma\delta}}{8} I_2^{\text{FDR}}, \quad (4.18)
\end{aligned}$$

with  $I_0^{\text{FDR}}$  and  $I_2^{\text{FDR}}$  given in eqs. 4.3 and 4.4. Finally, odd-rank vacuum tensor integrals, such as

$$\int [d^4 q] \frac{q^{\alpha_1} q^{\alpha_2} \dots q^{\alpha_{(2n+1)}}}{\bar{D}^m}, \quad (4.19)$$

vanish due to Lorentz invariance. Eqs. 4.18 are the gauge preserving conditions for one-loop tensor integrals [12] which corroborate the consistency of the FDR approach.

Depending on the way a calculation is performed, additional FDR integrals with powers of  $\mu$  in the numerator may arise, which should be treated accordingly to the *global treatment* prescription. For example

$$\int [d^4 q] \frac{\mu^2}{\bar{D}^3}. \quad (4.20)$$

should be considered as a logarithmically divergent one. FDR then requires an expansion of  $\frac{1}{\bar{D}^3}$  as in eq. 4.15, giving rise to

$$\int [d^4 q] \frac{\mu^2}{\bar{D}^3} = M^2 \lim_{\mu \rightarrow 0} \mu^2 \int d^4 q \left( \frac{1}{\bar{D}^3 \bar{q}^2} + \frac{1}{\bar{D}^2 \bar{q}^4} + \frac{1}{\bar{D} \bar{q}^6} \right) \Big|_{\mu=\mu_R} = \frac{i\pi^2}{2}. \quad (4.21)$$

Analogously

$$\begin{aligned}
\int [d^4 q] \frac{\mu^2}{\bar{D}^2} &= M^4 \lim_{\mu \rightarrow 0} \mu^2 \int d^4 q \left( \frac{2}{\bar{D} \bar{q}^6} + \frac{1}{\bar{D}^2 \bar{q}^4} \right) \Big|_{\mu=\mu_R} = i\pi^2 M^2, \\
\int [d^4 q] \frac{q^2 \mu^2}{\bar{D}^4} &= M^2 \lim_{\mu \rightarrow 0} \mu^2 \int d^4 q q^2 \left( \frac{1}{\bar{D}^4 \bar{q}^2} + \frac{1}{\bar{D}^3 \bar{q}^4} + \frac{1}{\bar{D}^2 \bar{q}^6} + \frac{1}{\bar{D} \bar{q}^8} \right) \Big|_{\mu=\mu_R} = \frac{i\pi^2}{3}, \\
\int [d^4 q] \frac{\mu^4}{\bar{D}^4} &= M^2 \lim_{\mu \rightarrow 0} \mu^4 \int d^4 q \left( \frac{1}{\bar{D}^4 \bar{q}^2} + \frac{1}{\bar{D}^3 \bar{q}^4} + \frac{1}{\bar{D}^2 \bar{q}^6} + \frac{1}{\bar{D} \bar{q}^8} \right) \Big|_{\mu=\mu_R} = -\frac{i\pi^2}{6}.
\end{aligned} \tag{4.22}$$

Eqs. 4.21 and 4.22 can be proved by direct integration. However, it is more elegant to observe that finite contributions may arise only when  $\mu^2$  and  $\mu^4$  hit  $1/\mu^2$  and  $1/\mu^4$  poles, respectively, which are more easily extracted by reinserting back eq. 4.15. Therefore

$$\begin{aligned}
\int [d^4 q] \frac{\mu^2}{\bar{D}^3} &= -\mu^2 \int d^4 q \frac{1}{\bar{q}^6} \\
\int [d^4 q] \frac{\mu^2}{\bar{D}^2} &= -2M^2 \int d^4 q \frac{1}{\bar{q}^6} \\
\int [d^4 q] \frac{q^2 \mu^2}{\bar{D}^4} &= -\mu^2 \int d^4 q \frac{q^2}{\bar{q}^8} \\
\int [d^4 q] \frac{\mu^4}{\bar{D}^4} &= -\mu^4 \int d^4 q \frac{1}{\bar{q}^8},
\end{aligned} \tag{4.23}$$

which reproduce the expected result. Finally, by power counting

$$\begin{aligned}
\int [d^4 q] \frac{\mu^{2j}}{\bar{D}^k} &= 0 \quad \text{when } k > 2 + j, \\
\int [d^4 q] \frac{\mu^{2j+1}}{\bar{D}^k} &= 0.
\end{aligned} \tag{4.24}$$

It is quite remarkable the perfect parallelism between eqs. 4.21, 4.22 and 4.24 and their counterparts in DR (see [13, 14])

$$\begin{aligned}
\int [d^4 q] \frac{\mu^2}{\bar{D}^3} &= - \int d^n q \frac{\tilde{q}^2}{\bar{D}^3} \\
\int [d^4 q] \frac{\mu^2}{\bar{D}^2} &= - \int d^n q \frac{\tilde{q}^2}{\bar{D}^2} \\
\int [d^4 q] \frac{q^2 \mu^2}{\bar{D}^4} &= - \int d^n q \frac{q^2 \tilde{q}^2}{\bar{D}^4} \\
\int [d^4 q] \frac{\mu^4}{\bar{D}^4} &= \int d^n q \frac{\tilde{q}^4}{\bar{D}^4} \\
\int [d^4 q] \frac{\mu^{2j}}{\bar{D}^k} &= \int d^n q \frac{(\tilde{q}^2)^j}{\bar{D}^k} = 0 \quad \text{when } k > 2 + j,
\end{aligned} \tag{4.25}$$

where  $\tilde{q}^2$  is the  $\epsilon$ -dimensional part of  $q^2$ .

With the help of these results, properties such as <sup>14</sup>

$$\begin{aligned}
\int [d^4 q] \frac{\bar{q}^2 - M^2}{\bar{D}^3} &= \int [d^4 q] \frac{1}{\bar{D}^2}, \\
\int [d^4 q] \frac{(\bar{q}^2 - M^2)^2}{\bar{D}^4} &= \int [d^4 q] \frac{1}{\bar{D}^2}, \\
\int [d^4 q] \frac{\bar{q}^2 - M^2}{\bar{D}^2} &= \int [d^4 q] \frac{1}{\bar{D}}, \\
\int [d^4 q] \frac{\bar{q}^2 - M^2}{\bar{D}} &= \int [d^4 q] = 0,
\end{aligned} \tag{4.26}$$

follow, which guarantee that all usual manipulations are allowed in the FDR integrands. As a by-product, the Passarino-Veltman [15], the OPP reduction approaches [16, 17], together with all available literature to compute the contributions generated by the appearance of  $\mu$  in the numerator [18–24], can also be used in FDR.

## 5 Two loops and beyond

As a two-loop example, consider the integral

$$I^{\text{FDR}} = \int [d^4 q_1][d^4 q_2] \frac{1}{\bar{D}_1 \bar{D}_2 \bar{D}_{12}}, \tag{5.1}$$

where

$$\begin{aligned}
\bar{D}_1 &= \bar{q}_1^2 - m_1^2, \\
\bar{D}_2 &= \bar{q}_2^2 - m_2^2, \\
\bar{D}_{12} &= \bar{q}_{12}^2 - m_{12}^2
\end{aligned} \tag{5.2}$$

and  $q_{12} \equiv q_1 + q_2$ . Identity 2.8 can be used to rewrite

$$\begin{aligned}
\frac{1}{\bar{D}_1 \bar{D}_2 \bar{D}_{12}} &= \left[ \frac{1}{\bar{q}_1^2 \bar{q}_2^2 \bar{q}_{12}^2} \right] \\
&+ \frac{m_1^2}{(\bar{D}_1 \bar{q}_1^2) \bar{q}_2^2 \bar{q}_{12}^2} + \frac{m_2^2}{\bar{q}_1^2 (\bar{D}_2 \bar{q}_2^2) \bar{q}_{12}^2} + \frac{m_{12}^2}{\bar{q}_1^2 \bar{q}_2^2 (\bar{D}_{12} \bar{q}_{12}^2)} \\
&+ \frac{m_1^2 m_2^2}{(\bar{D}_1 \bar{q}_1^2) (\bar{D}_2 \bar{q}_2^2) \bar{q}_{12}^2} + \frac{m_1^2 m_{12}^2}{(\bar{D}_1 \bar{q}_1^2) \bar{q}_2^2 (\bar{D}_{12} \bar{q}_{12}^2)} + \frac{m_2^2 m_{12}^2}{\bar{q}_1^2 (\bar{D}_2 \bar{q}_2^2) (\bar{D}_{12} \bar{q}_{12}^2)} \\
&+ \frac{m_1^2 m_2^2 m_{12}^2}{(\bar{D}_1 \bar{q}_1^2) (\bar{D}_2 \bar{q}_2^2) (\bar{D}_{12} \bar{q}_{12}^2)}.
\end{aligned} \tag{5.3}$$

The term between square brackets is, as usual, a vacuum integrand, which extracts the overall quadratic UV divergence of  $I^{\text{FDR}}$ . The following three produce logarithmically divergent (sub)-integrals and the last four can be integrated in four dimensions. The next step is singling out the remaining divergences. By rewriting

$$\frac{1}{\bar{q}_{12}^2} = \frac{1}{\bar{q}_2^2} - \frac{q_1^2 + 2(q_1 \cdot q_2)}{\bar{q}_2^2 \bar{q}_{12}^2}, \tag{5.4}$$

---

<sup>14</sup>They can also be directly proved from the identities in eq. 4.15.



**Figure 8.** Two-loop (left) and one-loop (right) logarithmically divergent subtraction vacuum scalar integrands. Dots denote propagator squared and  $\mu$  is the unphysical mass running in each line.

one obtains

$$\frac{m_1^2}{(\bar{D}_1 \bar{q}_1^2) \bar{q}_2^2 \bar{q}_{12}^2} = m_1^2 \left[ \frac{1}{\bar{q}_1^4 \bar{q}_2^2 \bar{q}_{12}^2} \right] + \frac{m_1^4}{(\bar{D}_1 \bar{q}_1^4)} \left[ \frac{1}{\bar{q}_2^4} \right] - m_1^4 \frac{q_1^2 + 2(q_1 \cdot q_2)}{(\bar{D}_1 \bar{q}_1^4) \bar{q}_2^4 \bar{q}_{12}^2}, \quad (5.5)$$

where the two vacuum integrands extract overall logarithmic and overlapping logarithmic sub-divergences, respectively <sup>15</sup>, and the last term is integrable. The remaining two integrands in the second line of eq. 5.3 can be treated analogously.  $I^{\text{FDR}}$  then reads

$$\begin{aligned} I^{\text{FDR}} \equiv & \lim_{\mu \rightarrow 0} \int d^4 q_1 \int d^4 q_2 \left( \frac{m_1^2 m_2^2}{(\bar{D}_1 \bar{q}_1^2)(\bar{D}_2 \bar{q}_2^2) \bar{q}_{12}^2} + \frac{m_1^2 m_{12}^2}{(\bar{D}_1 \bar{q}_1^2) \bar{q}_2^2 (\bar{D}_{12} \bar{q}_{12}^2)} + \frac{m_2^2 m_{12}^2}{\bar{q}_1^2 (\bar{D}_2 \bar{q}_2^2) (\bar{D}_{12} \bar{q}_{12}^2)} \right. \\ & - m_1^4 \frac{q_1^2 + 2(q_1 \cdot q_2)}{(\bar{D}_1 \bar{q}_1^4) \bar{q}_2^4 \bar{q}_{12}^2} - m_2^4 \frac{q_2^2 + 2(q_1 \cdot q_2)}{\bar{q}_1^4 (\bar{D}_2 \bar{q}_2^4) \bar{q}_{12}^2} - m_{12}^4 \frac{q_{12}^2 - 2(q_1 \cdot q_{12})}{\bar{q}_1^4 \bar{q}_2^2 (\bar{D}_{12} \bar{q}_{12}^4)} \\ & \left. + \frac{m_1^2 m_2^2 m_{12}^2}{(\bar{D}_1 \bar{q}_1^2)(\bar{D}_2 \bar{q}_2^2)(\bar{D}_{12} \bar{q}_{12}^2)} \right) \Big|_{\mu=\mu_R}. \end{aligned} \quad (5.6)$$

Other divergent two-loop integrals, such as

$$\int [d^4 q_1][d^4 q_2] \frac{1}{\bar{D}_1^2 \bar{D}_2 \bar{D}_{12}}, \quad (5.7)$$

can be obtained by derivation with respect to masses.

As already observed, quadratically divergent vacuum integrands, such as the first term in eq. 5.3, do not contribute when  $\mu \rightarrow 0$ , and one is left with the only two possible logarithmically divergent subtraction scalar (sub)-diagrams shown in figure 8, which are of the type of those appearing in eq. 5.5. Therefore, besides eq. 4.6, a two-loop correspondence holds

$$\frac{1}{\epsilon^2} + \text{UC subtraction after integration} \quad \leftrightarrow \quad \frac{1}{\bar{q}_1^4 \bar{q}_2^2 \bar{q}_{12}^2} \text{ subtraction before integration.} \quad (5.8)$$

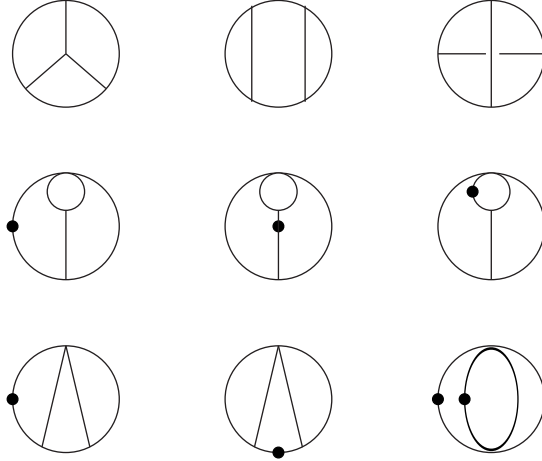
Only Universal Constants appear again in the left part of eq. 5.8 because of the existence of only one possible subtraction term.

At three loop the situation is more involved because nine irreducible topologies (see figure 9) may generate logarithmically divergent vacuum diagrams. In this case, FDR and DR might start differing diagram by diagram, although the gauge invariance properties of the FDR integral guarantees the equivalence of the two approaches <sup>16</sup>.

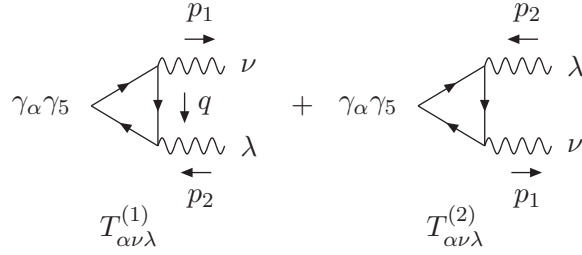
<sup>15</sup>The structure of one-loop counterterm naturally appears for the latter.

<sup>16</sup>See section 7 for a more detailed discussion on this point.





**Figure 9.** Irreducible three-loop topologies giving rise to logarithmically divergent subtraction vacuum diagrams. Dots denote propagators squared.



**Figure 10.** The two diagrams generating the ABJ anomaly.

## 6 The ABJ anomaly and $\gamma_5$

In this section, I reproduce the known ABJ anomaly [25, 26] with the FDR approach. Two massless fermion loop diagrams contribute, as shown in figure 10. When contracted with  $p = p_1 - p_2$ , the first term gives

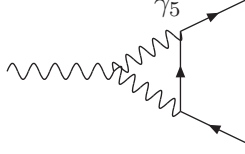
$$p^\alpha T_{\alpha\nu\lambda}^{(1)} = -i \frac{e^2}{(2\pi)^4} \int [d^4 q] \text{Tr} \left[ \not{p} \gamma_5 \frac{1}{Q_2 - \mu} \gamma_\lambda \frac{1}{Q_0 - \mu} \gamma_\nu \frac{1}{Q_1 - \mu} \right], \quad (6.1)$$

where  $Q_i = q + p_i$  ( $p_0 = 0$ ) and all massless propagators are shifted by  $\mu$ , as required by FDR. Rewriting

$$\not{p} = (Q_1 - \mu) - (Q_2 - \mu) \quad (6.2)$$

produces

$$\begin{aligned} p^\alpha T_{\alpha\nu\lambda}^{(1)} = & -i \frac{e^2}{(2\pi)^4} \int [d^4 q] \left( \text{Tr} \left[ \gamma_5 \frac{1}{Q_2 - \mu} \gamma_\lambda \frac{1}{Q_0 - \mu} \gamma_\nu \right] - \text{Tr} \left[ \gamma_5 \frac{1}{Q_{-1} - \mu} \gamma_\nu \frac{1}{Q_0 - \mu} \gamma_\lambda \right] \right. \\ & \left. + 2\mu \text{Tr} \left[ \gamma_5 \frac{1}{Q_2 - \mu} \gamma_\lambda \frac{1}{Q_0 - \mu} \gamma_\nu \frac{1}{Q_1 - \mu} \right] \right), \end{aligned} \quad (6.3)$$



**Figure 11.** Example of diagram in which the  $\mu$  dependence cannot be fully reabsorbed into the fermion masses.

where a shift  $q \rightarrow q - p_1$ <sup>17</sup> has been performed in the second trace, and  $Q_{-1} \equiv q - p_1$ . The contribution of the second diagram is obtained by replacing  $p_1 \leftrightarrow -p_2$  and  $\lambda \leftrightarrow \nu$ , thus all terms not proportional to  $\mu$  drop in the sum  $T = T^{(1)} + T^{(2)}$

$$p^\alpha T_{\alpha\nu\lambda} = -i \frac{e^2}{4\pi^4} \text{Tr}[\gamma_5 \not{p}_2 \gamma_\lambda \gamma_\nu \not{p}_1] \int [d^4 q] \mu^2 \frac{1}{\bar{D}_0 \bar{D}_1 \bar{D}_2}, \quad (6.4)$$

where  $\bar{D}_i = (q + p_i)^2 - \mu^2$ . The FDR integral is given in eq. 4.21, leading to the correct answer

$$p^\alpha T_{\alpha\nu\lambda} = \frac{e^2}{8\pi^2} \text{Tr}[\gamma_5 \not{p}_2 \gamma_\lambda \gamma_\nu \not{p}_1]. \quad (6.5)$$

Finally, I comment on the role of  $\gamma_5$ . In closed fermion loops, the FDR interpretation of  $\mu$  as a propagator shift allows one to fully reabsorb it into the fermion masses, providing unambiguous results also in the presence of  $\gamma_5$ . This was actually done in the previous calculation, where the result in eq. 6.5 could also be obtained by starting with a physical fermion mass  $m_f$  and replacing  $m_f \rightarrow m_f + \mu$  in the *finite* contribution proportional to  $m_f^2$ . However, in open fermion chains where not all  $\not{q}$  are linked to fermion masses, the ambiguity of *when* shifting  $\not{q} \rightarrow \not{q} - \mu$  is potentially present. For example, in the combination

$$(\cdots \not{q} \gamma_5 \frac{1}{\not{q}} \cdots)$$

occurring in the diagram of figure 11, the operations of anticommuting  $\gamma_5$  and shifting  $\not{q}$  do not commute. This is solved, as in DR, by considering the chiral theory as a good, gauge invariant starting point [27], which means, in practice, that  $\gamma_5$  must be anticommutated towards the external spinors *before* shifting  $\not{q}$ .

## 7 Renormalization

Since the UV infinities are subtracted right from the beginning, there is no need, in FDR, to add counterterms to the Lagrangian. One can indeed prove that

$$G_{\ell-loop}^{\text{FDR}}(\mu_R) = G_{\ell-loop}^{\text{DR}}(\mu_R), \quad (7.1)$$

---

<sup>17</sup>Legal in FDR.

where  $G^{\text{FDR}}$  is a generic  $\ell$ -loop Green function computed in FDR,  $G^{\text{DR}}$  the same Green function calculated in DR, but after renormalization<sup>18</sup>, and  $\mu_R$  is the renormalization scale. The generic form of  $G^{\text{FDR}}$  is

$$G_{\ell\text{-loop}}^{\text{FDR}}(\mu_R) = \sum_{i=0}^{\ell} a_i^{\text{DR}} \log^i(\mu_R) + R^{\text{DR}}(\{p, M\}) + R_0, \quad (7.2)$$

where  $R^{\text{DR}}(\{p, M\})$  is a term depending on the kinematical variables of the process and  $R_0$  an  $a$ -dimensional constant. By construction, the coefficients  $a_i^{\text{DR}}$  and  $R^{\text{DR}}(\{p, M\})$  are the same one would compute in DR. In fact, the logarithmic dependence on  $\mu_R$  is fully included in the definition of FDR integral, and any kinematical information is only contained in the finite part  $J_F$  of eq. 3.7, which is common to both DR and FDR. The only possible difference is  $R_0$ , because FDR requires to entirely drop the constant parts of the logarithmically divergent integrals, while DR only infinities and Universal Constants. However, one proves, by contradiction, that also  $R_0$  is common in the two schemes. If FDR misses some part of  $R_0$ , it could be fixed back by enforcing the Ward-Slavnov-Taylor identities of the theory, which requires to compute them in FDR to be able to adjust, by hand, all terms which violate them. But since FDR *is*, by construction, gauge invariant, no violation would be found, therefore eq. 7.1 is proven.

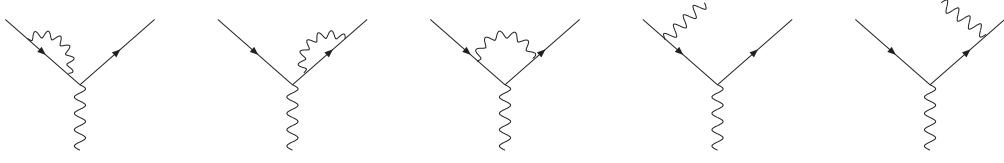
If  $G_{\ell\text{-loop}}^{\text{FDR}}$  belongs to a renormalizable theory, expressing its free parameters in terms of observables trades the unphysical scale  $\mu_R$  for a physical one, ensuring the equivalence of FDR with any other consistent renormalization scheme. Particularly interesting is the case of non-renormalizable field theories, such as quantum gravity. Measuring the parameters of the Lagrangian does not guarantee any longer the disappearance of  $\mu_R$ . However,  $G_{\ell\text{-loop}}^{\text{FDR}}$  is finite in four dimensions and eq. 7.2 still holds, so that one additional measurement can be used, in principle, to fix  $\mu_R$  order by order, making the theory predictive<sup>19</sup>. A possible interpretation is that the original theory, possibly due to non perturbative effects, or to a very complicate structure of its vacuum, could be not complete enough, or not to allow, a full determination of its UV counterpart, as happens in renormalizable theories. If it is the case, eq. 7.2 could still provide a parametrization, in terms of  $\mu_R$ , of unknown unphysical phenomena which do not decouple. Measuring  $\mu_R$ , definitively integrates out all unphysical degrees of freedom, leaving the observable spectrum free of UV effects. Whether this is a valid way out, is debatable. The important points to keep in mind are that

- the Lagrangian is left untouched;
- gauge invariance is not broken;
- four-dimensionality is kept;
- one additional measurement is enough to fix  $\mu_R$  at any perturbative order.

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<sup>18</sup>I assume the same renormalized parameters in  $G^{\text{DR}}$  and  $G^{\text{FDR}}$ .

<sup>19</sup>This is allowed because the limit  $\mu \rightarrow 0$  is fully taken in  $G_{\ell\text{-loop}}^{\text{FDR}}$ , and, as a result, the original  $\mu$  is replaced by  $\mu_R$ , which does not need to be small.



**Figure 12.** Virtual and real diagrams contributing to  $Z \rightarrow f\bar{f}$ .

## 8 Infrared and collinear divergences

In massless theories, the FDR insertion of  $\mu$  in the propagators (eq. 3.5) naturally regulates infrared and collinear divergent loop integrals. Unitarity requires a mass  $\mu$  for the external massless particles as well. Thus, one expects the  $\mu$  dependence to cancel, in the limit  $\mu \rightarrow 0$ , when adding virtual and real contributions. In this section, I verify that this happens by computing the  $\mathcal{O}(\alpha)$  QED corrections to the decay rate  $\Gamma(Z \rightarrow f\bar{f})$  with massless fermions. It provides an explicit example of FDR calculation in which all three types of divergences (UV, infrared and collinear) are simultaneously present.

The contributing diagrams are shown in figure 12. The virtual part is computed very much as in the DR case, but the tensor reduction should be performed in four dimensions and the integrals interpreted in FDR. The needed loop functions are

$$\begin{aligned}
B(s) &= \int [d^4q] \frac{1}{(q^2 - \mu^2)((q+p)^2 - \mu^2)} \Big|_{p^2=s}, \\
B_0 &= \int [d^4q] \frac{1}{(q^2 - \mu^2)(q^2 + 2(q \cdot p))} \Big|_{p^2=\mu^2}, \\
B_1 &= \frac{1}{p^2} \int [d^4q] \frac{(q \cdot p)}{(q^2 - \mu^2)(q^2 + 2(q \cdot p))} \Big|_{p^2=\mu^2}, \\
B'_0 &= \frac{d}{dp^2} B_0 \Big|_{p^2=\mu^2}, \\
B'_1 &= \frac{d}{dp^2} B_1 \Big|_{p^2=\mu^2}, \\
C(s) &= \int [d^4q] \frac{1}{(q^2 - \mu^2)(q^2 + 2(q \cdot p_1))(q^2 - 2(q \cdot p_2))} \Big|_{p_1^2=p_2^2=\mu^2; (p_1+p_2)^2=s}, \\
C_R &= \int [d^4q] \frac{\mu^2}{(q^2 - \mu^2)(q^2 + 2(q \cdot p_1))(q^2 - 2(q \cdot p_2))}, \tag{8.1}
\end{aligned}$$

and are listed in appendix B. The total virtual contribution reads

$$\Gamma_V(Z \rightarrow f\bar{f}) = \Gamma_0(Z \rightarrow f\bar{f}) \frac{\alpha}{\pi} \left[ -\frac{1}{2} \ln^2 \left( \frac{\mu^2}{s} \right) - \frac{3}{2} \ln \left( \frac{\mu^2}{s} \right) + \frac{7}{18} \pi^2 + \frac{\pi}{2\sqrt{3}} - \frac{7}{2} \right], \tag{8.2}$$

where  $\Gamma_0(Z \rightarrow f\bar{f})$  is the tree level result.

In the computation of the real part, all final state particles have a common mass  $\mu$  and the amplitude squared is integrated over the 3-body phase-space, parametrized as

$$\int d\Phi_3 = \frac{\pi^2}{4s} \int ds_{12} ds_{23}, \quad (8.3)$$

where  $\sqrt{s}$  is the center of mass energy of the decaying particle and  $s_{ij}$  are two of the three possible final state two-body invariant masses. It is convenient to introduce the variables

$$x = \frac{s_{12} - \mu^2}{s}, \quad z = \frac{s_{23} - \mu^2}{s}, \quad (8.4)$$

in terms of which the relevant bremsstrahlung integrals are

$$\begin{aligned} I_1 &= \int_R dx dz \frac{1}{x^2} = \int_R dx dz \frac{1}{z^2}, \\ I_2 &= \int_R dx dz \frac{1}{xz}, \\ I_3 &= \int_R dx dz \frac{1}{x} = \int_R dx dz \frac{1}{z}, \\ I_4 &= \int_R dx dz \frac{x}{z} = \int_R dx dz \frac{z}{x}, \end{aligned} \quad (8.5)$$

where  $R$  is the full available phase-space. They are reported in appendix B, and give

$$\Gamma_V(Z \rightarrow f\bar{f}) = \Gamma_0(Z \rightarrow f\bar{f}) \frac{\alpha}{\pi} \left[ \frac{1}{2} \ln^2 \left( \frac{\mu^2}{s} \right) + \frac{3}{2} \ln \left( \frac{\mu^2}{s} \right) - \frac{7}{18} \pi^2 - \frac{\pi}{2\sqrt{3}} + \frac{17}{4} \right]. \quad (8.6)$$

By summing eqs. 8.2 and 8.6 one obtains

$$\Gamma_{\text{TOT}}(Z \rightarrow f\bar{f}) = \Gamma_0(Z \rightarrow f\bar{f}) \left( 1 + \frac{3}{4} \frac{\alpha}{\pi} \right), \quad (8.7)$$

which is the expected result.

In eq. 8.6 I used, for the photon polarization vectors

$$\sum_{pol} \epsilon_\alpha \epsilon_\beta^* = -g_{\alpha\beta}. \quad (8.8)$$

Care must be taken in more complicated cases to verify that the missing pieces do not contribute when  $\mu \rightarrow 0$ .

## 9 Conclusions

The FDR approach discriminates between observable physics and unobservable infinities occurring at large values of the integration momenta. This interpretation allows one to define, in a mathematical consistent way and at any order in the perturbative expansion, the UV divergent integrals appearing in quantum field theories as four-dimensional integrals over the physical spectrum only. The physics of renormalizable theories is reproduced and

the possibility for the non-renormalizable theories to become predictive is opened. Infrared and collinear divergences can also be naturally accommodated.

FDR looks promising for realistic calculations too. No counterterms need to be added to the Lagrangian, and both virtual and real contributions are kept in four dimensions, which may be particularly interesting at two (or more) loops, where subtracting powers of  $1/\epsilon$  can become cumbersome in DR. These more practical aspects will be investigated in the near future.

## A 1- and 2-point one-loop scalar integrals in FDR

In this appendix, I explicitly compute the FDR scalar integrals  $I_0^{\text{FDR}}$  and  $I_2^{\text{FDR}}$  of eqs. 4.3 and 4.4. By means of a common Feynman parametrization one rewrites

$$K \equiv \int d^4q \left( \frac{1}{\bar{D}^2 \bar{q}^2} + \frac{1}{\bar{D} \bar{q}^4} \right) = \Gamma(3) \int_0^1 d\alpha \int d^4q \frac{1}{(q^2 - M^2 \alpha - \mu^2)^3}, \quad (\text{A.1})$$

where  $\bar{D} = (\bar{q}^2 - M^2)$  and  $\bar{q}^2 = q^2 - \mu^2$ . Performing the integral in  $d^4q$  gives

$$K = -i\pi^2 \int_0^1 d\alpha \frac{1}{M^2 \alpha + \mu^2} = -\frac{i\pi^2}{M^2} \ln \frac{M^2 + \mu^2}{\mu^2}, \quad (\text{A.2})$$

from which eq. 4.3 immediately follows. Analogously,

$$\begin{aligned} \int d^4q \frac{1}{\bar{D} \bar{q}^4} &= \Gamma(3) \int_0^1 d\alpha (1 - \alpha) \int d^4q \frac{1}{(q^2 - M^2 \alpha - \mu^2)^3} \\ &= -i\pi^2 \int_0^1 d\alpha \frac{(1 - \alpha)}{M^2 \alpha + \mu^2} = \frac{i\pi^2}{M^4} \left( M^2 - (M^2 + \mu^2) \ln \frac{M^2 + \mu^2}{\mu^2} \right), \end{aligned} \quad (\text{A.3})$$

which gives eq. 4.4. The same results can be obtained via eq. 3.9, by expressing  $I_0^{\text{FDR}}$  and  $I_2^{\text{FDR}}$  as the difference of two DR integrals.

## B Virtual and real integrals for $\Gamma(Z \rightarrow f \bar{f})$

The one-loop scalar integrals appearing in the computation of  $\Gamma_Z$  are (see eq. 8.1)

$$\begin{aligned} B(s) &= i\pi^2 \left[ \ln \left( -\frac{\mu^2 - i\epsilon}{s} \right) + 2 \right], \\ B_0 &= -i\pi^2 \left( \frac{\pi}{\sqrt{3}} - 2 \right), \\ B_1 &= -\frac{1}{2} B_0, \\ B'_0 &= \frac{i\pi^2}{\mu^2} \left( \frac{2}{3} \frac{\pi}{\sqrt{3}} - 1 \right), \\ B'_1 &= -\frac{1}{2} B'_0, \\ C(s) &= \frac{i\pi^2}{s} \left[ \frac{1}{2} \ln^2 \left( -\frac{\mu^2 - i\epsilon}{s} \right) + \frac{\pi^2}{9} \right], \\ C_R &= \frac{i\pi^2}{2}. \end{aligned} \quad (\text{B.1})$$

As for the real part, the integrals in eq. 8.5 read

$$\begin{aligned}
I_1 &= \frac{s}{\mu^2} \left( \frac{2}{3} \frac{\pi}{\sqrt{3}} - 1 \right) \\
I_2 &= \frac{1}{2} \ln^2 \left( \frac{\mu^2}{s} \right) - \frac{7}{18} \pi^2 \\
I_3 &= -\ln \left( \frac{\mu^2}{s} \right) - 1 - \frac{\pi}{\sqrt{3}} \\
I_4 &= -\frac{3}{4} - \frac{1}{2} \ln \left( \frac{\mu^2}{s} \right) - \frac{\pi}{2\sqrt{3}}.
\end{aligned} \tag{B.2}$$

All terms which vanish in the limit  $\mu \rightarrow 0$  are neglected in eqs. B.1 and B.2.

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